THEONA

Numeric-Analytical Satellite Theory with its Applications and Special Functions

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Reasons

The numeric-analytical (semi-analytical) theory THEONA was developed in the 1980s at Keldysh Institute of Applied Mathematics (KIAM) *as rapid and efficient orbit propagator* for various space problems associated with activities of the Ballistic Center KIAM including orbit prediction for flights of the orbital station, manned (Soyuz), and automatic (Progress) spaceships.

Calculations based on THEONA approximately 100 times faster than analogous numerical integration with retention of high accuracy.

THEONA's tools work with satellite orbits in large range of altitudes (120 to 35000 km)

Ballistic Center KIAM

The Ballistic Center of Keldysh Institute of Applied Mathematics (KIAM) works during more 50 years. First leader was academician, professor Efraim Lazarevitch Akim. Now the leader of Ballistic Center KIAM is professor Andrey Georgievitch Tuchin.

Principal activities of the Ballistic Center KIAM:

- ballistic and navigation flight control (orbital station, manned and automatic spaceships, small probes, interplanetary missons)
- mission design and mission analysis
- Space debris Center (catalogue, observations)
- on-board software

Basic Principles

- ✓ *Intermediary orbit* based on the Generalized Problem of 2 Fixed Centers
- ✓ *"Step-by-step" method of orbit propagation*
- ✓ *Osculating elements* of intermediary orbit
- ✓ *Analytical integration* within the "step" of propagation
- ✓ *Special functions of orbit elements* with recurrence formulae
- ✓ *All essential perturbing forces*
- ✓ *Orbit corrections*
- ✓ *Formation Flying:* "Virtual chief" with deviations of "assistants"
- ✓ *Satellite Constellations:* "Virtual network" (and "meshes") with deviations of satellite positions
- ✓ *Phase Beam (of trajectories)*

Intermediary Orbit

- ✓ *Generalized Problem of 2 Fixed Centers (GP2FC)* [Aksenov,Grebenikov,Demin;1958]

$$U_{\oplus} = \frac{\mu(1+i\sigma)/2}{\sqrt{x^2+y^2+\left[z-c(\sigma+i)\right]^2}} + \frac{\mu(1-i\sigma)/2}{\sqrt{x^2+y^2+\left[z-c(\sigma-i)\right]^2}} =$$

$$= \frac{\mu}{r}\cdot\left[1 - J_2\left(\frac{r_e}{r}\right)^2 P_2\left(\frac{z}{r}\right) - J_3\left(\frac{r_e}{r}\right)^3 P_3\left(\frac{z}{r}\right) - \sum_{n=4}^{\infty}\frac{\gamma_n}{r^n} P_n\left(\frac{z}{r}\right)\right]$$

$$\text{where } \gamma_n = -\frac{1}{2}(c/r_e)^n \cdot \left[(1+i\sigma)\cdot(\sigma+i)^n + (1-i\sigma)\cdot(\sigma-i)^n\right], \quad \gamma_4 \approx 0.72\cdot J_4$$

- ✓ *Osculating Eulerian elements:* $\mathbf{q} = \{a, e_x = e\cos\omega, e_y = e\sin\omega, \cos i, \Omega, T_\Omega\}$
- ✓ *Variable of integration: argument of latitude* $u = v + \omega$
- ✓ *HEO:* $\mathbf{q} = \{a, e, \omega, \cos i, \Omega, T_\pi\}$ *and true anomaly* $v = u - \omega$
- ✓ *NEO:* $\mathbf{q} = \{a, e\cos(\omega+\Omega), e\sin(\omega+\Omega), \sin(i/2)\cos\Omega, \sin(i/2)\sin\Omega, T_\lambda\}$

  and $u = v + \Omega$

Orbit propagation

- *"Step-by-step" method* with sequential summation of perturbations of *osculating* Eulerian elements:

$$\mathbf{q}^{(\mathrm{step}+1)} = \mathbf{q}^{(\mathrm{step})} + \Delta\mathbf{q}^{(\mathrm{step})}$$

- *Perturbations* $\Delta\mathbf{q}^{(\mathrm{step})}$ within each "step" are calculated analytically by using *Special functions of orbit elements* with recurrence formulae

- *Osculating elements:*
  1) Direct correlation between Eulerian orbital elements and the state vectors (without any additional matching)
  2) Large range of orbit altitudes *[120 km; 200000 km]*
  3) non-conservative forces
  4) changes solar and geomagnetic activities, pole motion, …
  5) renewals of orbit parameters (tracking, corrections, etc.)

Force model

- ✓ *All essential perturbing forces:*
  - the Earth's (or central celestial body) gravitational potential model (arbitrary order and degree of harmonics)
  - atmospheric drag forces
    (different dynamical models of atmospheric density)
    (GOST 25645.115-1990, GOST R 25645.166-2004, NRLMSISE00, DTM77, etc.)
  - third body gravity forces (with JPL DE 405 ephemerides)
  - solar radiation pressure (including shadow effects)
  - tidal forces (only 2nd Love number)
- ✓ *Forces induced by maneuvers (for active satellites):*
  - impulses
  - long-term constant thrust
  - low thrust

Magnitudes

- *Magnitudes of Perturbing Forces:*
  - Non-central forces of GP2FC $O(\gamma)$

    ($J_2 \sim 10^{-3}$, $J_3 \sim 2\cdot10^{-6}$ for Earth; $J_2 \sim 2\cdot10^{-3}$, $J_3 \sim 2.5\cdot10^{-5}$ for Mars)
    ($J_2 \sim 1.2 \div 1.6\cdot10^{-2}$ for Jupiter, Saturn, Uran; $J_2 \sim 2\cdot10^{-4}$ for Moon)
  - Essential conservative forces $O(\beta)$

    (the rest of spherical harmonics, 3rd body gravity, etc.)
  - Atmospheric drag forces $O(\alpha)$

    ($\sim 10^{-5}$ for low altitudes)
- *Accuracy of THEONA:*
  - Version 1 (old): $\Delta\mathbf{q} \sim O\left(\gamma^3, \gamma\beta, 0.005\cdot\alpha\right) \sim 10^{-9}$ (for Earth)
  - Version 2 (new): $\Delta\mathbf{q} \sim O\left(\gamma^2\beta, \beta^2, \gamma^2\alpha, \alpha^2\right)$, $\Delta a \sim O\left(\gamma^2\beta, \beta^2, \gamma^2\alpha, \gamma\alpha^2, \alpha^3\right)$

Special Functions

- ✓ *Inclination functions*
  in conservative perturbations (gravity, solar radiation pressure, tidal)
  - *Principal functions*
    (differ by factors from the Jeffreys' inclination functions)
  - *Supplementary functions*
    (to include 3rd order terms)
- ✓ *Functions of 2 arguments* depending on *mean motion & excentricity*
  in conservative perturbations (gravity, solar radiation pressure, tidal)
  to integrate perturbating functions on latitude argument
- ✓ *Associated Legendre functions* with non-integer degree
  to integrate perturbations due to air drag taking into account:
  – the height scale's changes,
  – the "afternoon hump" of atmospheric density

Inclination Functions

- *Principle functions:*

$$Q_{mk}^{n}(\cos I)=\sqrt{\frac{(n+m)!}{(n-m)!}\cdot\frac{(n-k)!}{(n+k)!}}\cdot P_{n}^{k}(0)\cdot i^{n-m}P_{mk}^{n}(\cos I)$$

where

$$P_{mk}^{n}(\cos I)=\frac{1}{2\pi}\int_{0}^{2\pi}\left(\cos\frac{I}{2}+\sin\frac{I}{2}\cdot \boldsymbol{e}^{i\varphi}\right)^{n+k}\left(\cos\frac{I}{2}+\sin\frac{I}{2}\cdot \boldsymbol{e}^{-i\varphi}\right)^{n-k}\boldsymbol{e}^{i(m-k)\varphi}d\varphi$$

The transformation to the unimodular unitary (2×2) rotation group SU(2)

- *Supplementary functions:*

$$\bar{Q}_{mk}^{n}(\cos I)=\sqrt{\frac{(n+m)!}{(n-m)!}\cdot\frac{(n-k)!}{(n+k)!}}\cdot P_{n}^{k+1}(0)\cdot i^{n-m}P_{mk}^{n}(\cos I)$$

Inclination Functions (cont.)

- ✓ *Principle functions:*

$$Q_{mk}^{n}(\zeta)=\begin{cases}\sqrt{\dfrac{(n+m)!}{(n-m)!}\cdot\dfrac{(n-k-1)!!\,(n+k-1)!!}{(n-k)!!\,(n+k)!!}}\cdot i^{k-m}P_{mk}^{n}(\zeta) & ,\quad (n-k)-even\\ 0 & ,\quad (n-k)-odd\end{cases}$$

$$Q_{nk}^{n}(\zeta)=\frac{(2n)!}{(n-k)!(n+k)!}\cdot\left(\frac{1+\zeta}{2}\right)^{\frac{n+k}{2}}\cdot\left(\frac{1-\zeta}{2}\right)^{\frac{n-k}{2}},\quad (n-k)-even$$

- ✓ *Supplementary functions:*

$$\bar{Q}_{mk}^{n}(\zeta)=\begin{cases}0 & ,\quad (n-k)-even\\ \sqrt{\dfrac{(n+m)!}{(n-m)!}\cdot\dfrac{(n-k)!!\,(n+k)!!}{(n-k-1)!!\,(n+k-1)!!}}\cdot i^{k-m}P_{mk}^{n}(\zeta) & ,\quad (n-k)-odd\end{cases}$$

Inclination Functions (cont.)

✓ *Recurrence relations:*

$$Q_{mk}^{n}(\cos I) = \frac{n+m-1}{n-m} \cdot Q_{mk}^{n-2} + \frac{2n-1}{2(n-m)} \cdot \sin I \cdot \left[Q_{m,k+1}^{n-1} - Q_{m,k-1}^{n-1} \right], \quad if\ \ m < n$$

$$Q_{nk}^{n}(\cos I) = \frac{2n-1}{2} \cdot \left[(1-\cos I) \cdot Q_{n-1,k+1}^{n-1} + (1+\cos I) \cdot Q_{n-1,k-1}^{n-1} \right], \quad if\ \ m = n$$

$$\bar{Q}_{mk}^{n}(\cos I) = \frac{(n-k-1)(n+k-1)}{(n-k)(n+k)} \cdot \frac{n+m-1}{n-m} \cdot \bar{Q}_{mk}^{n-2} + \\ + \frac{2n-1}{2(n-m)} \cdot \sin I \cdot \left[\frac{n-k-1}{n-k} \cdot \bar{Q}_{m,k+1}^{n-1} - \frac{n+k-1}{n+k} \cdot \bar{Q}_{m,k-1}^{n-1} \right]$$

$$\bar{Q}_{nk}^{n}(\cos I) = \frac{2n-1}{2\left(k^{2}+1\right)} \cdot \left[(1-\cos I) \cdot \bar{Q}_{n-1,k+1}^{n-1} - (1+\cos I) \cdot \bar{Q}_{n-1,k-1}^{n-1} \right], \quad if\ \ m = n$$

Special Functions of 2 Arguments

- *Integral representation:*

$$\mathcal{G}_m^n(x,e) = \frac{1}{2\pi}\int_0^{2\pi}(1+e\cos\psi)^n \cdot \boldsymbol{e}^{ix(M-\psi)} \cdot \boldsymbol{e}^{im\psi}\,d\psi$$

- *Unfinity scalar product of the Bessel functions and the Jacobi functions:*

$$\mathcal{G}_m^n(x,e) = \sum_{k=-\infty}^{\infty} J_k(xe)\cdot \mathfrak{P}^n_{m-x,k-x}\left(\frac{1}{\sqrt{1-e^2}}\right)$$

where the Jacobi functions:

$$\mathfrak{P}^n_{mk}(\mathrm{ch}\,\tau) = \frac{1}{2\pi}\int_0^{2\pi}\left(\cosh\frac{\tau}{2}+\sinh\frac{\tau}{2}\cdot\boldsymbol{e}^{i\varphi}\right)^{n+k}\left(\cosh\frac{\tau}{2}+\sinh\frac{\tau}{2}\cdot\boldsymbol{e}^{-i\varphi}\right)^{n-k}\boldsymbol{e}^{i(m-k)\varphi}\,d\varphi$$

The transformation to the unimodular quasi-unitary (2×2) rotation group QU(2)

- *The Hansen coefficients are a particular case:*

$$X^{(k)}_{n,m} = \left(1-e^2\right)^{n+3/2}\cdot\mathcal{G}_{k-m}^{-(n+2)}(k,e)$$

Functions $\mathcal{G}_m^n(x,e)$

- *Addition theorem:*

$$\mathcal{G}_k^{n+m}(x+y,e) = \sum_{j=-\infty}^{\infty} \mathcal{G}_j^n(x,e) \cdot \mathcal{G}_{k-j}^m(y,e)$$

- *Symmetry property:*

$$\mathcal{G}_m^n(-x,e) = \mathcal{G}_{-m}^n(-x,e)$$

- *Particular case:*

$$\mathcal{G}_m^n(0,e) = \left(1-e^2\right)^{n/2} \mathfrak{P}_{m0}^n\left(\frac{1}{\sqrt{1-e^2}}\right) = \left(1-e^2\right)^{n/2} P_n^m\left(\frac{1}{\sqrt{1-e^2}}\right) \quad if \quad n \geq 0$$

- *Expansion in terms of the Newcomb operators:*

$$\mathcal{G}_m^n = \left(1-e^2\right)^{n+1/2} \cdot \sum_{k=0}^{\infty} \Pi_m^{|m|+2k}(-n-2, x-m) \cdot e^{|m|+2k}$$

Functions $\mathcal{G}_m^n(x,e)$

- ✓ *Recurrence relations:*

$$\mathcal{G}_m^{n+j}(x,e)=\sum_{k=-\infty}^{\infty}\mathcal{G}_k^n(0,e)\cdot\mathcal{G}_{m-k}^j(x,e)=\left(1-e^2\right)^{n/2}\sum_{k=-n}^{n}P_n^k\left(\frac{1}{\sqrt{1-e^2}}\right)\cdot\mathcal{G}_{m-k}^j(x,e)$$

$$\mathcal{G}_m^n(x,e)=\left(1-e^2\right)^{n/2}\sum_{k=-n}^{n}P_n^k\left(\frac{1}{\sqrt{1-e^2}}\right)\cdot\mathcal{G}_{m-k}^0(x,e),\quad (j=0)$$

$$\mathcal{G}_m^{n+1}(x,e)=\mathcal{G}_m^n(x,e)+\frac{e}{2}\cdot\left[\mathcal{G}_{m+1}^n(x,e)+\mathcal{G}_{m-1}^n(x,e)\right],\quad (j=1)$$

- ✓ *Modified functions of 2 arguments:*

$$g_m^n(x,e)=e^{-|m|}\cdot\mathcal{G}_m^n(x,e);\qquad g_m^n(-x,e)=g_{-m}^n(x,e)$$

$$g_m^{n+1}(x,e)=g_m^n(x,e)+\frac{1}{2}\cdot\left[g_{m-1}^n(x,e)+e^2 g_{m+1}^n(x,e)\right]\quad if\ \ m\geq 1$$

$$g_0^{n+1}(x,e)=g_0^n(x,e)+\frac{1}{2}\cdot\left[g_1^n(x,e)+g_1^n(-x,e)\right]$$

Legendre Functions

- ✓ *Dynamical model of atmosphere density:*

$$\rho = \rho_0 \left(1 + \frac{h - h_0}{\nu H}\right)^{\nu} \cdot \left[\sum_{n=0}^{N} E_n h^n + \cos^n \frac{\varphi}{2} \cdot \sum_{m=0}^{M} F_m h^m\right]$$

- ✓ *Expansion for power law density:*

$$(1 - \beta \cos E)^{-\lambda} = \beta^{-\lambda} \sum_{m=0}^{\infty} (2 - \delta_{m0}) \cdot \frac{\Gamma(\lambda - m)}{\Gamma(\lambda)} \cdot \mathfrak{P}_{-\lambda}^{m}(\beta) \cos mE \,, \quad \beta = \frac{ae}{\nu H}$$

- ✓ *Expansion for “afternoon hump” effect:*

$$\cos^n \frac{\varphi}{2} = \left(\frac{1 + \cos\varphi_\sigma \cos(u - \lambda_\sigma)}{2}\right)^{n/2} =$$

$$= \left(\frac{\sin\varphi_\sigma}{2}\right)^{n/2} \sum_{m=0}^{\infty} (2 - \delta_{m0}) \frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + m + 1)} \cdot \mathfrak{P}_{n/2}^{m}(\operatorname{cosec}\varphi_\sigma) \cos m(u - \lambda_\sigma)$$

Applications of THEONA

- ✓ *Orbit prediction* (complete)
- ✓ *Long-term propagation* (complete)
- ✓ *Orbit lifetime* (done, prospective in some other projects)
- ✓ *Problems of optimal maneuvering* (realized for various projects)
- ✓ *Satellite constellations* (complete)
- ✓ *Formation flying* (complete)
- ✓ *Mission analysis of the project "Apogee-IPM"* (done for 1st stage for Ballistic Center KIAM, prospective method)
- ✓ *Space Debris* (prospective)

Orbit Propagation

- ***Problem statement:***

  Long-term propagation (Cauchy problem) with intermediary "stop-node"

  Initial data:
  - orbit elements (or the state vector) in the time, ballistic coefficient
  - parameters of planned maneuvers

  Possible additional information:
  - tables of indexes of solar and geomagnetic activities

- ***Method:***

  THEONA propagator computes SBI (standard ballistic information) at each required "stop-nodes". SBI are used to predefine discrete events (raising/setting with respect to a ground location, zones of radio visibility & observation, crossing of required function) numerically/analytically using "stop-nodes"

- ***Features:***

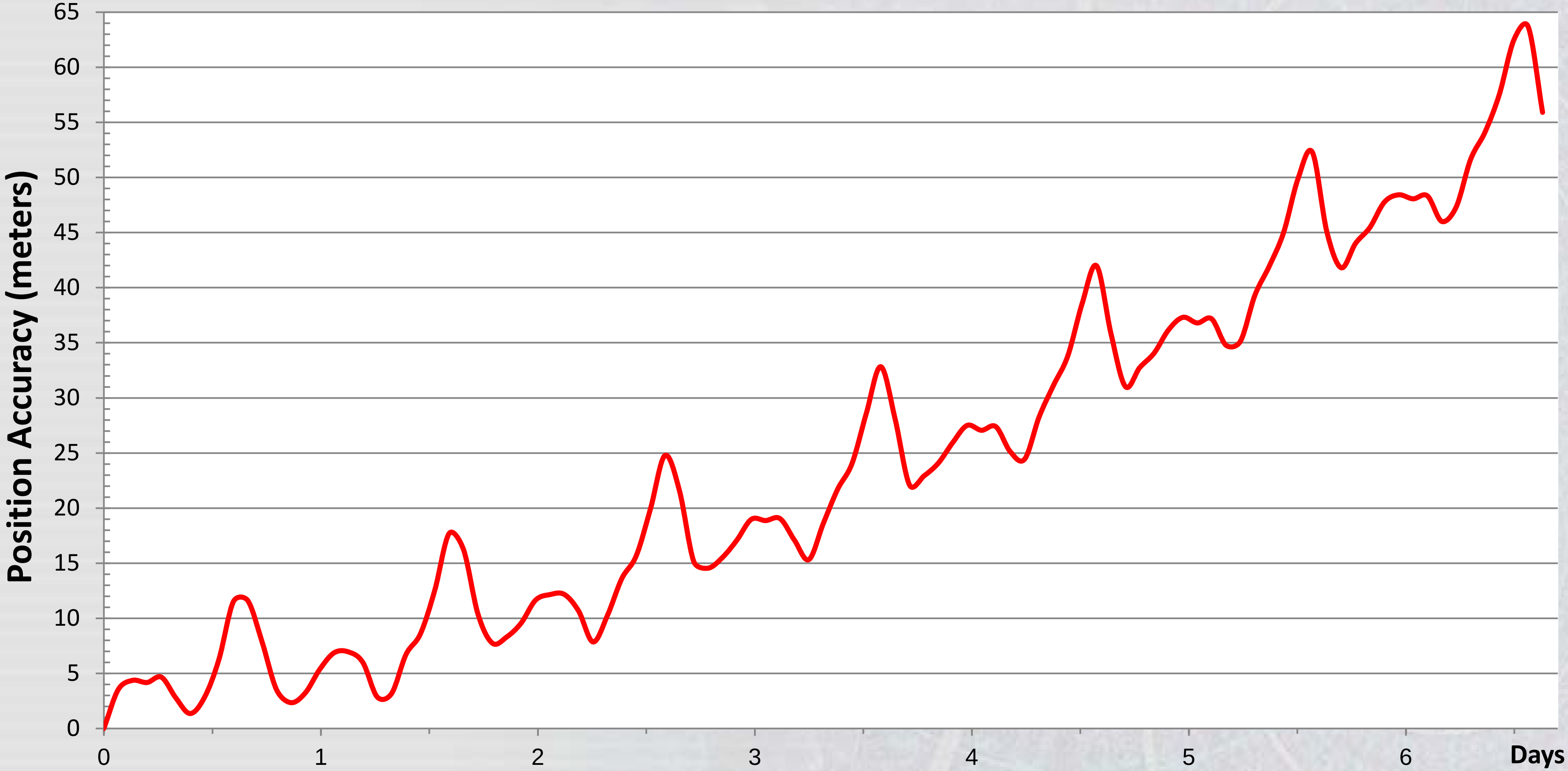
  Accuracy of THEONA's orbit propagation is caused by theory restrictions. The model *with concordant parameters* has better accuracy (as orbit prediction)

International Space Station

- ✓ *Orbit (GMT 2010/309/12:00:00):*

  Long-term propagation (ISS)

  Initial parameters: $a = 6762.435\,km,\ h = 342 \div 360\,km,\ e = 0.00105,\ i = 51.392\,deg$

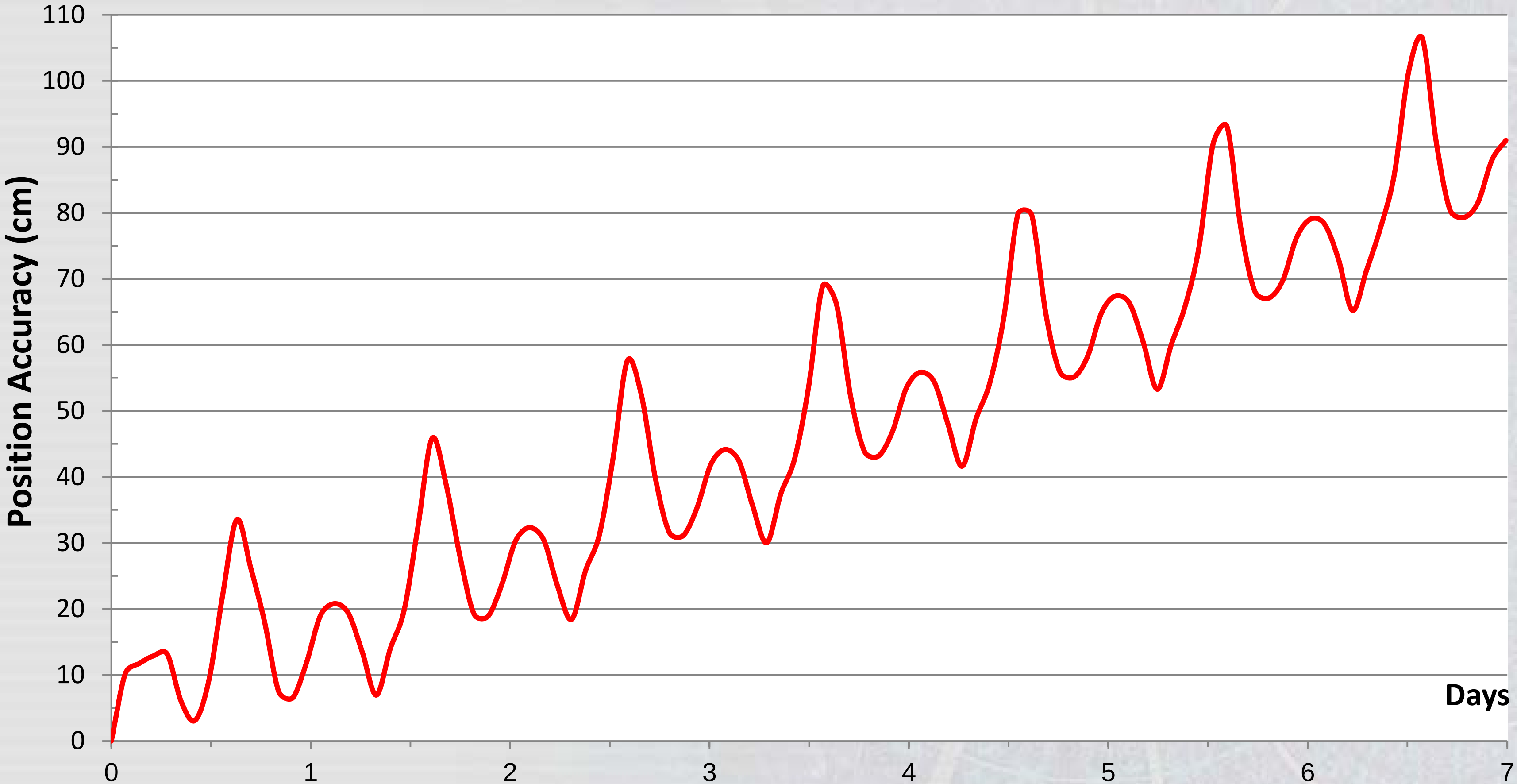


Starlette

- ✓ *Orbit (GMT 2011/222/15:55:25):*

  Long-term propagation (1975-010A, NORAD ID: 07646)

  Initial parameters: $a = 7344\,km,\ h = 812 \div 1114\,km,\ e = 0.0206,\ i = 49.825\,deg$

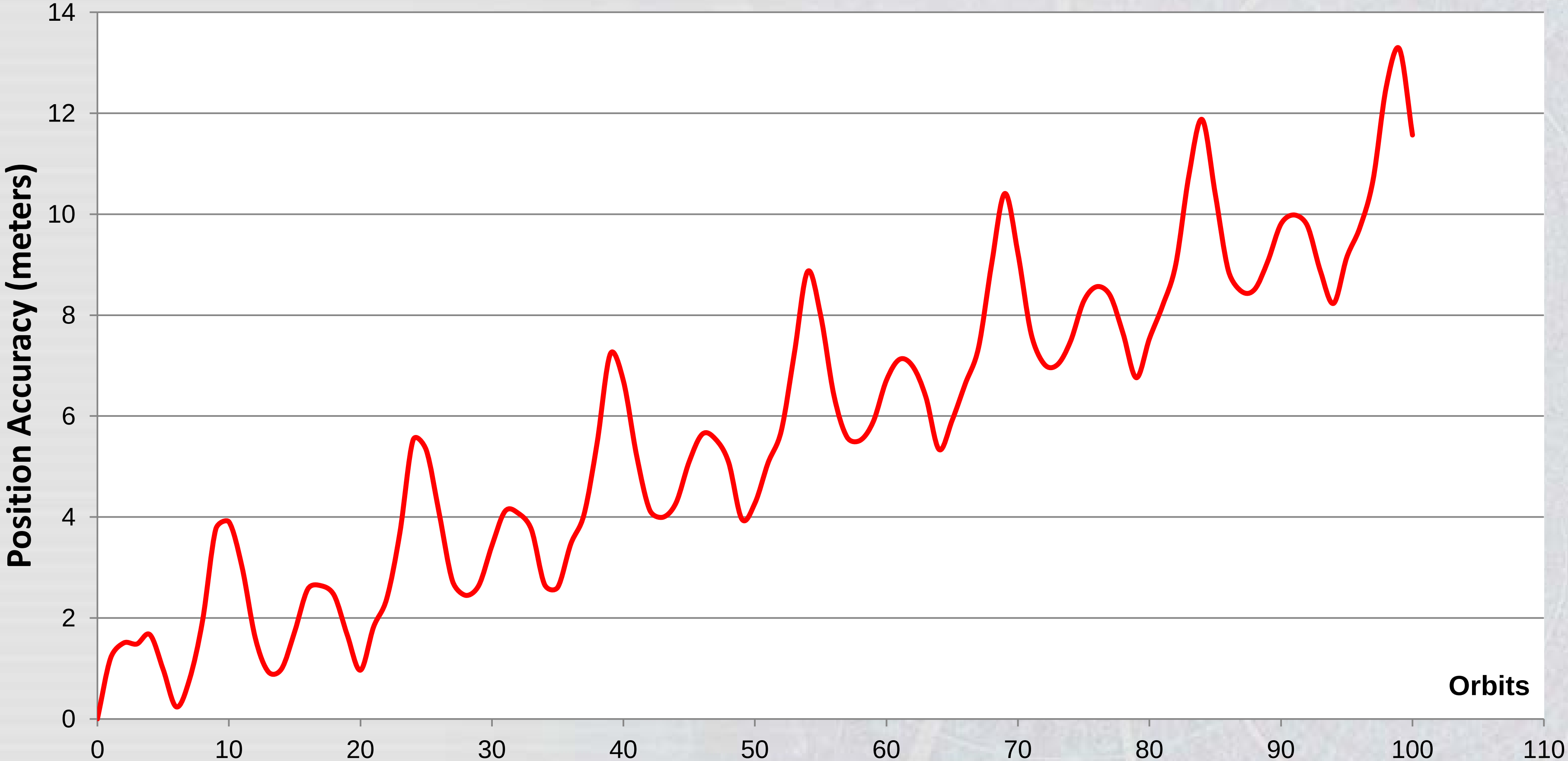


CanX-4 & 5

- ✓ *Orbit (GMT 2010/217/14:48:30):*

  Long-term propagation (PCO formation flying, distance = 100 *m*)

  Initial parameters: $a = 6920\,km,\ h \approx 550\,km,\ e = 0.001,\ i = 97.42\,deg$

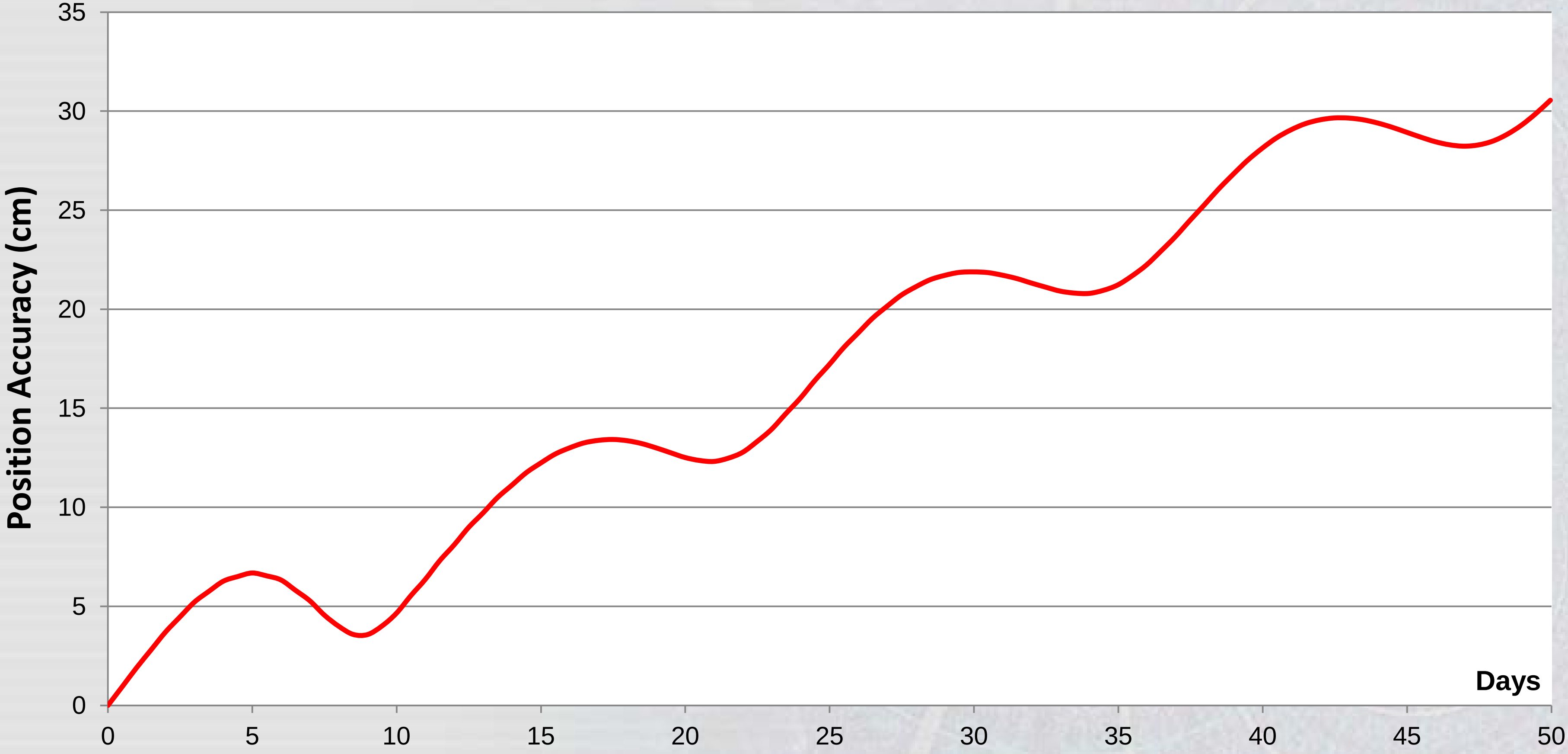


GPS

- ✓ *Orbit (GMT 2011/301/13:42:15):*

  Long-term propagation (NAVSTAR 66; 2011-036A, NORAD ID: 37753)

  Initial parameters: $a = 26560\,km,\ h = 20185 \div 20193\,km,\ e = 0.00016,\ i = 55\,deg$



Orbit Prediction

- ✓ ***Problem statement:***

  Orbit determination (based on data in short intervals)
  \+ *Orbit propagation* (on long future time interval)

- ✓ ***Method:***

  THEONA propagator with calculation of partial derivatives

- ✓ ***Example of Space Flight Control in the Ballistic Center of KIAM:***

  Regular long-term prediction of orbits of the Space Station "Mir" ("Salut-6"), manned ("Soyuz") and unmanned ("Progress") spacecrafts

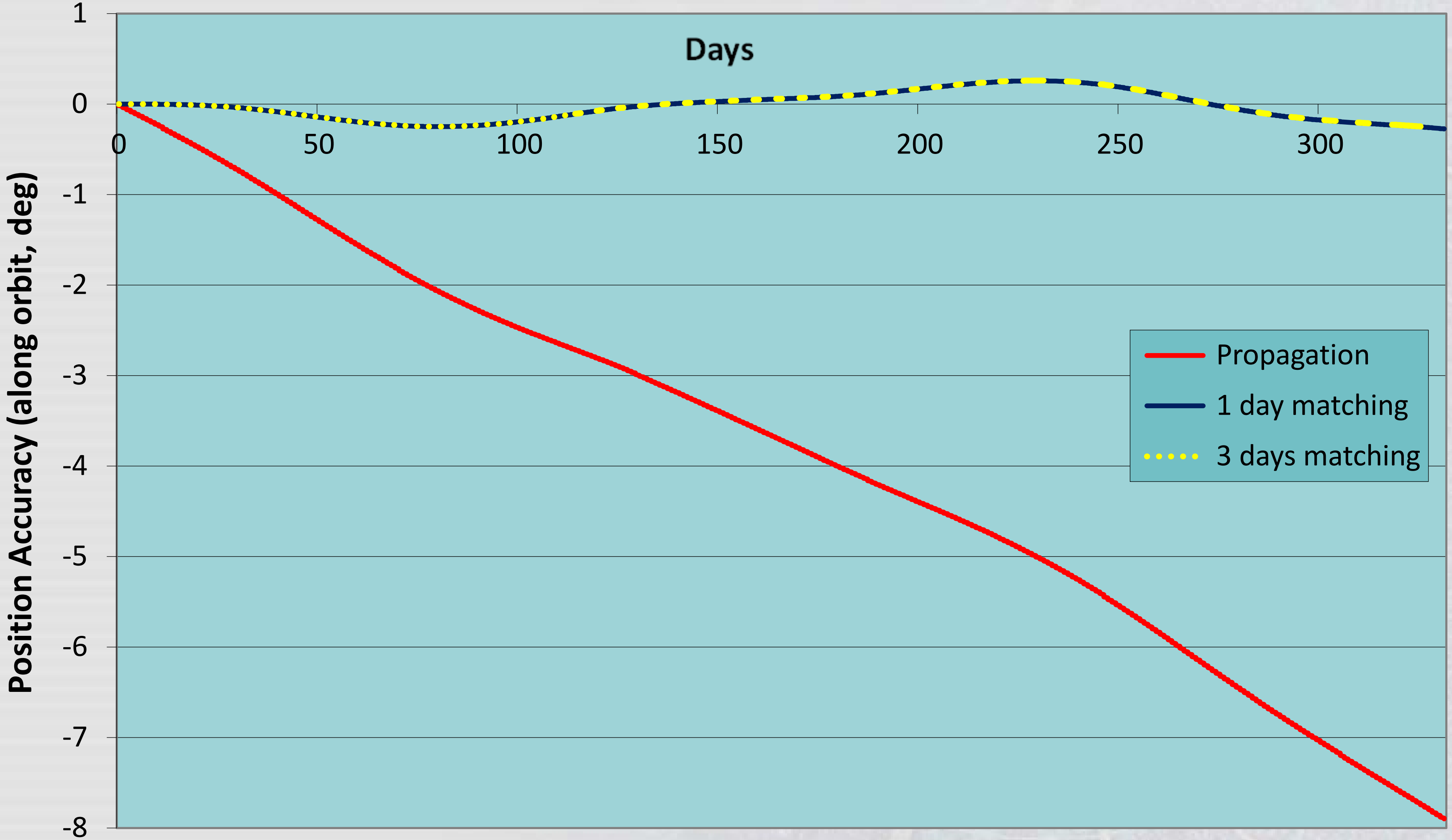
  Orbit determination taking into account data (measurements) in:
  - 3 contiguous revolutions (yesterday),
  - 3 contiguous revolutions (1 day later, today)

  Specified parameters:
  - orbit elements (or the state vector) in the time, ballistic coefficient,
  - maneuvers parameters (if they existed during this day)

  Propagation of determined orbit for 100 revolutions

Starlette

✓ *Orbit prediction (1 day or 3 days of determination) and propagation*



Orbit Lifetime

- ✓ ***Problem statement:***

  Initial data:
  - orbit elements (or the state vector) in the time, ballistic coefficient
  - minimum height (usually 120 km)

  Supplementary conditions for the problem of graveyard orbit:
  - required lifetime
  - area of the minimum existence

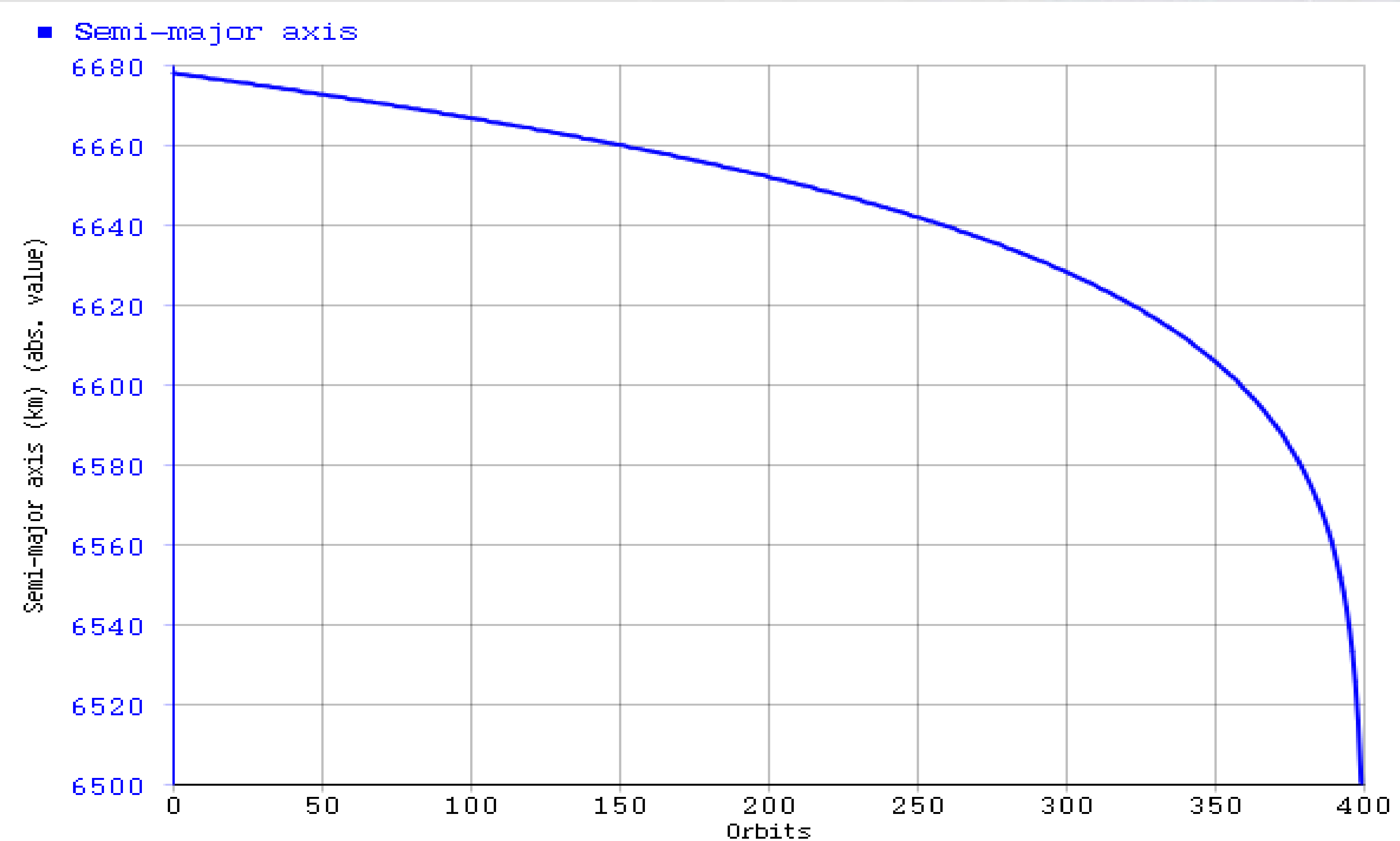
- ✓ ***Method:***

  Long-term propagation of orbit decreasing until it reaches the minimum height. In the problem of *graveyard orbit* THEONA propagator generates the beam of phase trajectories satisfying set conditions

- ✓ ***Remarks:***

  Solution of the problem of graveyard orbit is considered in collaboration with Dr. Andrey Baranov. There are prospective applications with method perfection.

Orbit Lifetime



Optimal Maneuvers

- ✓ *Problem statement:*

  Initial data:
  - orbit elements (or the state vector) in the time, ballistic coefficient
  - maneuver restrictions

- ✓ *Method:*

  Long-term propagation and short propagation. THEONA propagator takes into account multiple maneuvers as impulses, long-term propulsion, low thrust .

- ✓ *Remarks:*

  The problems of optimal maneuvering (different types) are efficiently solved in works of Dr. Andrey Baranov. He uses THEONA propagator (called TRACE) for two reasons: iterative schemes with rapid propagation, and the possibility to change numeric-analytical and numeric integration without additional procedures. Besides, TRACE propagator is applied in the problems of forming and keeping the satellite systems (satellite constellations and formation flying).

Satellite Formation Flying

✓ *Problem statement:*

Orbit propagation of absolute and relative motion of satellites

Initial data:

- orbit parameters of all satellites
- maneuvers to form and keep the formation

✓ *Method:*

THEONA propagates absolute motion of “virtual chief”. Partial derivatives permit to calculate relative motion of the satellites with good accuracy

✓ *Remarks:*

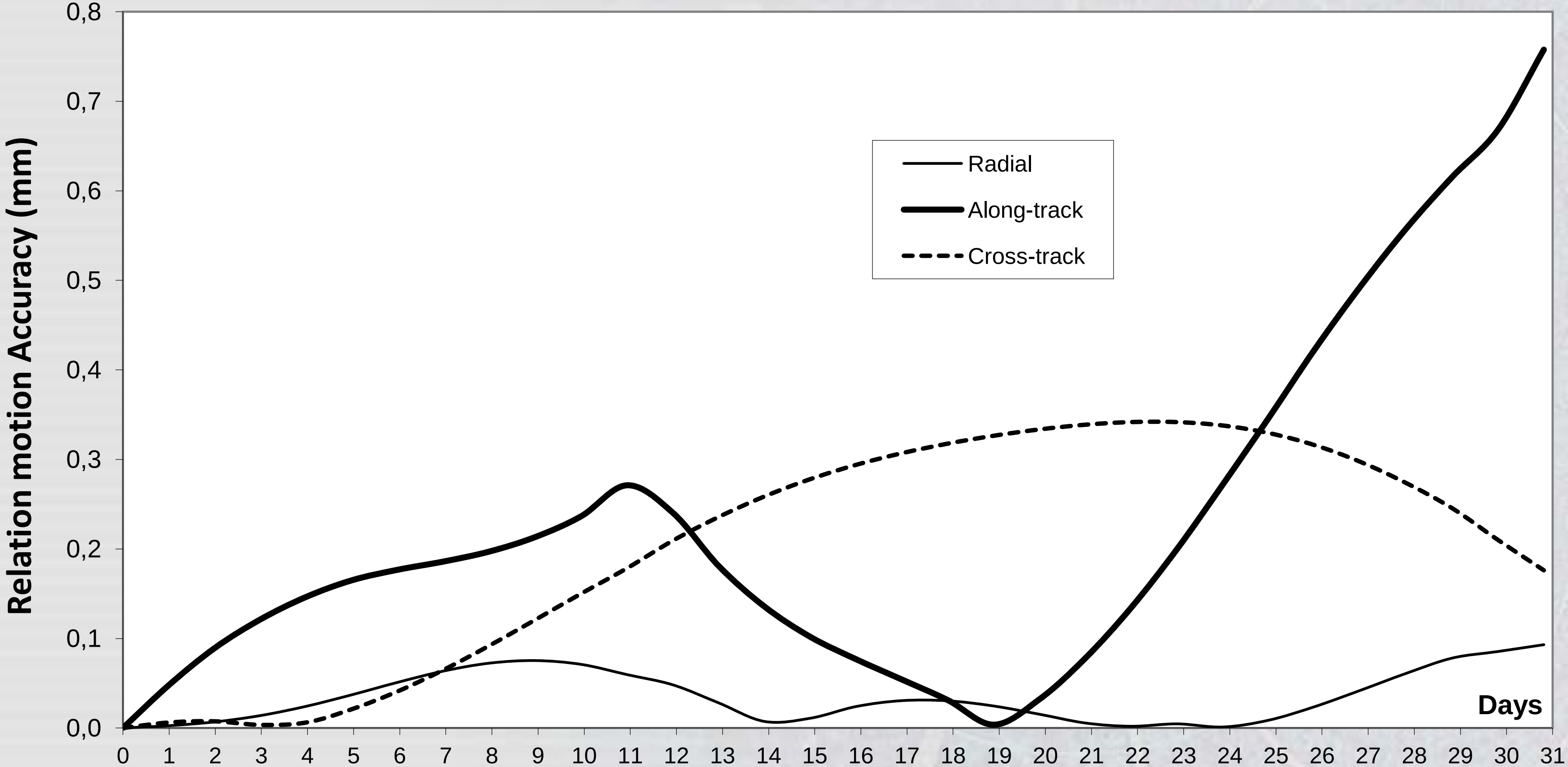
Method of THEONA has shown high efficiency for describing relative motion of satellites in the group. An accuracy depends on distances of satellites from “virtual chief”. If we consider more extended formation, efficiency of using partial derivatives decreases. At the moment prospective application of the method of phase beam is being developed.

CanX-4 & 5

- ✓ *Orbit (GMT 2010/217/14:48:30):*

Long-term propagation (PCO formation flying, distance = 100 *m*)

Initial parameters: $a = 6920\,km,\ h \approx 550\,km,\ e = 0.001,\ i = 97.42\,deg$



Satellite Constellations

- ✓ *Problem statement:*

  Orbit propagation of absolute and relative motion of satellites

  Initial data:
  - orbit parameters of all satellites
  - maneuvers to form and keep the constellation

- ✓ *Method:*

  THEONA propagates absolute motion of "virtual network" with "meshes". Using partial derivatives THEONA calculates relative motion of the satellites with high accuracy

- ✓ *Remarks:*

  Method of THEONA is efficient to describe relative motion of satellites in the constellation. At the moment prospective application of the method of phase beam is being developed. Future projects envisage a possibility to construct networks at various heights. This is prospective application for THEONA.

"Apogee-IPM"

- ✓ *Problem statement:*

  Search the orbit providing stable long-term passing of perigee over preset area

  Initial data:

  - orbit & satellite parameters (h_{min}=500 km, h_{max}=200000 km, i =56 deg)
  - coordinates and dimensions of the area (latitude of 52 deg North)
  - required (desirable) time interval (1 year)

- ✓ *Method:*

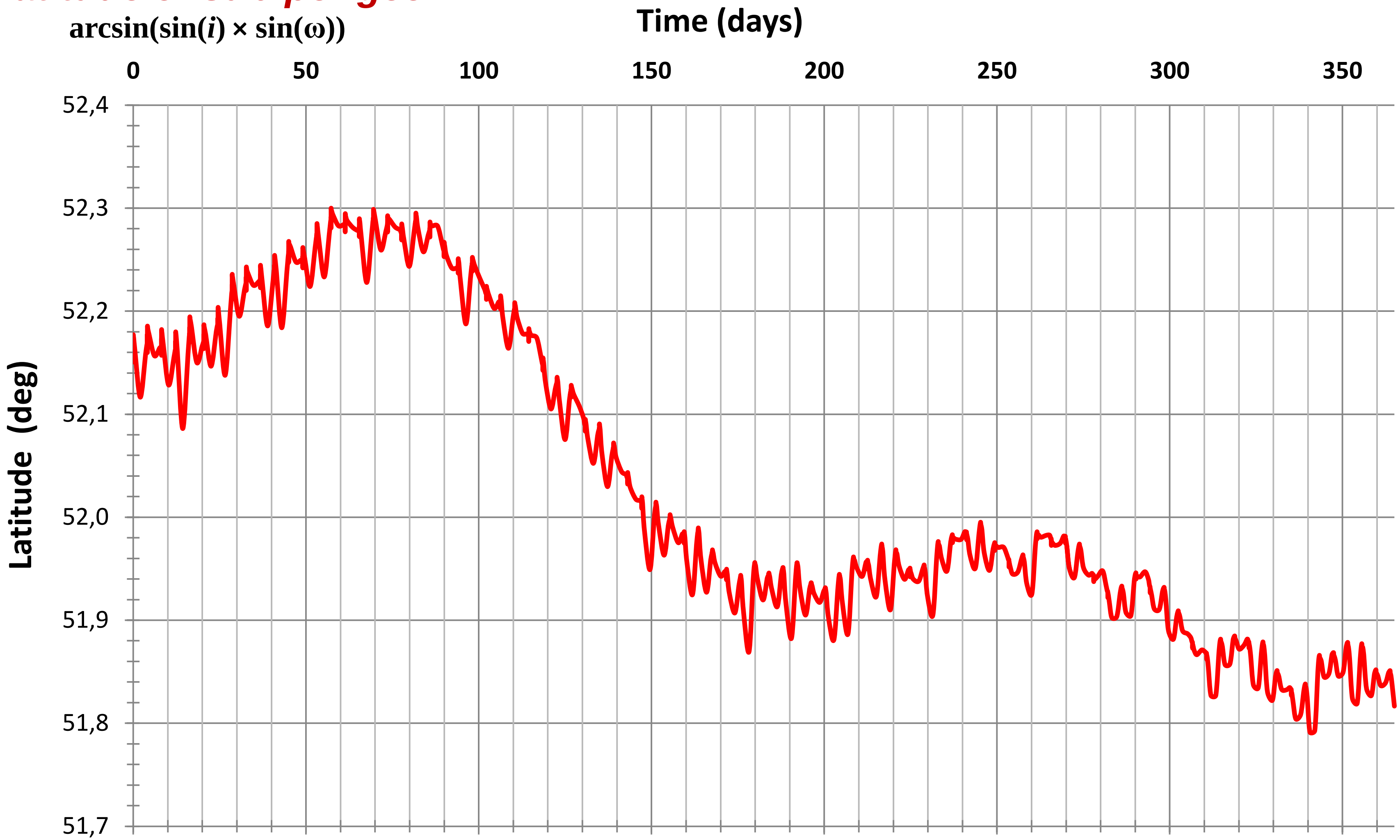
  An application uses the *method of phase beam*. THEONA-propagator calculates a set of phase trajectories. Phase changes are directed toward target orbit (by minimization of corresponding functional).

- ✓ *Remarks:*

  This prospective application is being developed. First realization was applied for the *project "Apogee-IPM"* in the Ballistic Center KIAM (leader Dr. Andrey Tuchin). As a result of studies orbits were found where the sub-perigee latitude remains nearly constant during 1 year. Other parameters give the solution.

"Apogee-IPM"

Latitude of sub-perigee



Space Debris

✓ *Problem statement:*

Space debris orbit propagation. Collision risk assessment

Initial data:

– orbit parameters of the satellite
– catalogue of space objects with orbit parameters

✓ *Method:*

An application uses the *method of phase beam*. THEONA-propagator calculates a set of phase trajectories. Collision risk is estimated by proximity of propagated tracks at short time intervals.

✓ *Remarks:*

The peculiarity of method is in use non-isochronous tracks with probabilistic distribution along trajectory. At the moment this method is under development and is being tested.

Thank you for attention

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